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$\mathcal{N}=1$ G_2 SYM theory and compactification to three dimensions

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Abstract: We study four dimensional $\mathcal{N}=2$ G_2 supersymmetric gauge theory on $R^3\times S^1$ deformed by a tree level superpotential. We will show that the exact superpotential can be obtained by making use of the Lax matrix of the corresponding integrable model which is the periodic Toda lattice based on the dual of the affine G_2 Lie algebra. At extrema of the superpotential the Seiberg-Witten curve typically factorizes, and we study the algebraic equations underlying this factorization. For U(N) theories the factorization was closely related to the geometrical engineering of such gauge theories and to matrix model descriptions, but here we will find that the geometrical interpretation is more mysterious. Along the way we give a method to compute the gauge theory resolvent and a suitable set of one-forms on the Seiberg-Witten curve. We will also find evidence that the low-energy dynamics of G_2 gauge theories can effectively be described in terms of an auxiliary hyperelliptic curve.

Keywords: Nonperturbative Effects, Supersymmetry and Duality, Supersymmetric Effective Theories.

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1. Introduction

Supersymmetric gauge theories have been in the center of attention for a long time. One of the reasons for this is that a large class of these theories, i.e. $\mathcal{N} = 1$ gauge theories, are likely to be of relevance for real world physics. The other reason is that non-supersymmetric gauge theories such as ordinary QCD can be considered as perturbations away from a supersymmetric point.

These theories have a rich structure and one can obtain exact results about their non-perturbative dynamics and hence about their vacuum structure (for a review, see e.g [1]). A major step towards the understanding of this structure was a general organizing principle put forward by Dijkgraaf and Vafa [2]. Motivated by earlier works [3]–[7] these authors have conjectured that the exact superpotential and gauge couplings for a wide class of $\mathcal{N} = 1$ gauge theories can be obtained by doing perturbative calculations in a dual matrix model only taking the planar diagrams into account. This conjecture has been verified using perturbative superspace techniques [8], and also using anomalies [9].

Despite the successes of this conjecture some of its features remain somewhat puzzling such as the distinguished role of the gluino condensate superfields, the appearance of the Veneziano-Yankielowicz superpotential and the capability of the matrix model approach to include all possible gauge theories.

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In trying to answer some of these questions, $\mathcal{N} = 2$ U(N) theories deformed by a $\mathrm{Tr}\, W(\phi)$ superpotential were considered on the space $R^3 \times S^1$[1] in [15] where, based on earlier works [16], it was conjectured that if the classical superpotential is $\mathrm{Tr}\, W(\phi)$ then the quantum superpotential will be just $\mathrm{Tr}\, W(M)$ where M is the Lax matrix of the integrable system that underlies the four dimensional theory.[2] In a consequent publication [17] the agreement of the vacuum structure obtained by the Lax matrix approach with the results obtained in four dimensions using the conventional field theoretic approach was proved for the gauge group U(N). In a separate work [18] the same result was proved using alternative methods. The above conjecture was tested for gauge groups SO/SP in [19] and again a complete agreement with the known results was shown. For some related comments see also [29].

If we replace classical groups by exceptional groups, several new questions arise. In [29] it was shown that the perturbative computation of the glueball superpotential described in [8] reduces to effectively zero-dimensional integrals even for exceptional groups. Therefore, one would expect that there still exists an appropriate notion of a matrix integral, however the meaning of the "planar diagrams" in such a matrix theory is not clear, nor is it known what replaces the Calabi-Yau geometry that was used to solve the matrix theory for the classical groups. Another issue is related to the ambiguity of the glueball superpotential also discussed in [29]. This ambiguity arises because the gauge theory with an arbitrary superpotential is not renormalizable, and to make it into a well-defined theory one needs to specify a suitable UV completion. This can in general be done in different ways, leading to different answers for terms of sufficiently high order in the glueball superfield. String theory prefers in some sense one particular UV completion, and a natural field-theoretic UV completion using embeddings in supergroups was described in [29]. This latter technique fails for exceptional groups, for which no natural field-theoretic UV completion is known.

To study these questions we consider in this paper the example of G_2, the exceptional group of lowest rank. We will find partial answers to the above questions. In particular, we will find some algebraic equations that in principle determine the geometry underlying the exceptional matrix models, though we were not able to put them in a nice form. We will also see that the Lax matrices provide a natural UV completion of gauge theories for all gauge groups, including exceptional ones, at least as far as holomorphic quantities are concerned.

The paper is organized as follows. In section 2 the classical description of $\mathcal{N} = 2$ SYM theory with gauge group G_2 is considered. Then the deformation by a tree level potential is studied. In section 3 the quantum description of $\mathcal{N} = 1$ G_2 four dimensional SYM is studied and the effective superpotential is computed.[3] The Lax matrix of the related integrable system is introduced and the different configurations where the Seiberg-Witten

[1] The compactification of the $\mathcal{N} = 2$ SYM theory to three dimensions was considered in [10]. For further discussions see for example [11]–[14].

[2] The relation between $\mathcal{N} = 2$ SYM theories and integrable system was discussed in several papers including [20]–[27]. For recent discussion in this direction and its relation with Dijkgraaf-Vafa conjecture see also [28].

[3] $\mathcal{N} = 1$ G_2 four dimensional SYM coupled to the different matters is also studied [31, 32].

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curve is factorized corresponding to the unbroken and broken gauge group cases as well as the superconformal field theory case are studied. In section 4 the theory on $R^3 \times S^1$ is considered and again the unbroken gauge symmetry case and the broken one are studied. In the former case the effective action in terms of the glueball field is obtained.The vacuum solutions are also interpreted from the view point of the corresponding integrable model. In section 5 we describe how the Lax matrix provides a UV completion and derive the corresponding gauge theory resolvent. We also consider the factorization problem and give a proof that enables us to state the extremization problem in purely algebraic terms. These equations should describe the geometry underlying the exceptional matrix model. In section 6 an argument is presented supporting the existence of a hyperelliptic curve for G_2. The last section is devoted to conclusion and remarks.

2. Classical description

In this section we review some classical aspects of the $\mathcal{N}=2$ SYM theory with gauge group G_2. The theory has a Coulomb branch where the gauge group is broken to $\mathrm{U}(1)^2$. The classical moduli space of the Coulomb branch is described by the characteristic polynomial

$$\mathcal{P}_{\text{class}}(x) = \frac{1}{x}\det(x\mathbf{1}-\phi) = x^6 - 2ux^4 + u^2x^2 - v\,, \tag{2.1}$$

where ϕ is the adjoint scalar component of the $\mathcal{N}=1$ chiral multiplet contained in the $\mathcal{N}=2$ vector multiplet. We assume in (2.1) that it takes values in the seven-dimensional fundamental representation of G_2, and using a gauge transformation and the equations of motion it can be assumed to take values in the Cartan subalgebra. The moduli parameters of the polynomial u and v are defined in terms of the gauge invariant parameters of the gauge group as follows

$$u = \frac{1}{2}u_2\,, \quad v = u_6 - \frac{1}{12}u_2^3\,, \qquad \text{where } u_k = \frac{1}{k}\,\mathrm{Tr}(\phi^k)\,. \tag{2.2}$$

The classical discriminant of the polynomial (2.1) up to a redundant numerical factor is

$$\Delta_{\text{class}} = -4u^3v + 27v^2\,, \tag{2.3}$$

whose zeroes give the points on the classical moduli space where the two of the zeroes of the polynomial coincide. Note that the classical discriminant is invariant under the following duality transformation

$$v \to -v + \frac{4}{27}u^3\,, \tag{2.4}$$

which reflects the fact that the root lattice of G_2 is self-dual.

Let us now consider a deformation of the theory given by adding a tree level superpotential given by

$$W_{\text{tree}} = g_2u + g_6v\,. \tag{2.5}$$

This deformation lifts most of the classical moduli space. To have a supersymmetric vacuum one needs to impose the D- and F-term equations. Taking ϕ to be diagonal implies that the

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D-term equation is satisfied, and for the F-term equations one should set $W' = 0$. More precisely, taking

$$\phi_{\rm class} = {\rm diag}(\phi_1+\phi_2, 2\phi_1-\phi_2, \phi_1-2\phi_2, 2\phi_2-\phi_1, \phi_2-2\phi_1, -\phi_1-\phi_2, 0) \tag{2.6}$$

the F-term condition reads

$$\begin{aligned}(\phi_2-2\phi_1)\left(g_2+2g_6(2\phi_2^4+2\phi_1^4+5\phi_2^3\phi_1-3\phi_2^2\phi_1^2-4\phi_2\phi_1^3)\right) &= 0\,,\\ (\phi_1-2\phi_2)\left(g_2+2g_6(2\phi_1^4+2\phi_2^4+5\phi_1^3\phi_2-3\phi_1^2\phi_2^2-4\phi_1\phi_2^3)\right) &= 0\,,\end{aligned} \tag{2.7}$$

which has two inequivalent solutions given by

$$\phi_1 = \phi_2 = 0 \tag{2.8}$$

and

$$\phi_1 = \phi_2 = \left(-\frac{g_2}{4g_6}\right)^{1/4}. \tag{2.9}$$

The first one corresponds to the case where the gauge group remains unbroken while the second one corresponds to the situation where the gauge group is broken to $\mathrm{SU}(2)\times\mathrm{U}(1)$ and in this case the classical superpotential is $W_{\rm tree} = g_2(-g_2/4g_6)^{1/2}$.

Using the explicit form of the gauge invariant parameters for these solutions one can see that the discriminant is also zero. Therefore the solutions correspond to the situation where the classical curve becomes degenerate. Explicitly one finds

$$\begin{aligned}\phi_1=\phi_2=0 \quad &\longrightarrow \quad \mathcal{P}_{\rm class} = x^6\\ \phi_1=\phi_2=e \quad &\longrightarrow \quad \mathcal{P}_{\rm class} = (x^2-e^2)^2(x^2-4e^2)\,,\end{aligned} \tag{2.10}$$

where $e = (-g_2/4g_6)^{1/4}$. Note that by making use of the duality transformation one could get a degenerate curve of the form $x^2(x^2-3e^2)^2$.

3. Quantum description

In this section we study the quantum aspects of $\mathcal{N}=1$ G_2 four dimensional SYM theory. In particular we shall compute the effective superpotential. In fact since our $\mathcal{N}=1$ theory can be thought of as an $\mathcal{N}=2$ theory deformed by a superpotential, one can use the exact result of $\mathcal{N}=2$ G_2 SYM theory to compute the quantum superpotential. Actually, one might suspect that the exact superpotential can be obtained from some kind of the Seiberg-Witten curve factorization, though, in the case of G_2 the curve is not hyperelliptic.

The Seiberg-Witten curve of G_2 gauge theory is given by the spectral curve of the periodic Toda chain based on the dual affine Lie algebra $G_2^{(1)}$ which is given in terms of twisted affine Lie algebra $D_4^{(3)}$ (see for example [33, page 511]). It is well known that the underlying integrable system, the Toda chain, admits a Lax pair for arbitrary gauge groups. Thus there exist two matrices M and A such that evolution of the theory can be described by the Lax equation

$$\frac{\partial M}{\partial t} = [M, A]\,. \tag{3.1}$$

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In our model the corresponding Lax matrix is given by

$$M=\begin{pmatrix} \phi_1+\phi_2 & y_2 & 0 & 0 & 0 & y_1 & -z & 0 \\ 1 & 2\phi_1-\phi_2 & 0 & ay_1 & by_1 & 0 & 0 & -z \\ 0 & 0 & \phi_1-2\phi_2 & -a & b & 0 & 0 & y_1 \\ 0 & a & -ay_2 & 0 & 0 & -a & ay_1 & 0 \\ 0 & b & by_2 & 0 & 0 & b & by_1 & 0 \\ 1 & 0 & 0 & -ay_2 & by_2 & 2\phi_2-\phi_1 & 0 & 0 \\ -\frac{y_0}{z} & 0 & 0 & a & b & 0 & \phi_2-2\phi_1 & y_2 \\ 0 & -\frac{y_0}{z} & 1 & 0 & 0 & 0 & 1 & -\phi_1-\phi_2 \end{pmatrix}, \tag{3.2}$$

where $a=\sqrt{1/2}$, $b=\sqrt{3/2}$ and there is a constraint on y_i given by $y_0y_2y_1^2=\Lambda^8/36$. The Seiberg-Witten curve is then obtained from the spectral curve $\det(x\mathbf{1}-M)=0$, which is

$$\mathcal{P}_{\text{quan}}(z,x):=3\left(z-\frac{\Lambda^8}{36z}\right)^2-x^2\left(z+\frac{\Lambda^8}{36z}\right)(6x^2-2u)-x^2P(x,u,v)=0\,, \tag{3.3}$$

where

$$P(x,u,v)=x^6-2ux^4+u^2x^2-v\,. \tag{3.4}$$

Here u and v are the moduli of the quantum curve which are functions of ϕ_i and y_i. Therefore the quantum moduli space of the Coulomb branch of G_2 theory is parameterized by u and v. These parameters can be given in terms of traces of the Lax matrix as follows

$$u=\frac{1}{2}U_2\,,\qquad v=U_6-\frac{1}{12}U_2^3+5U_2\left(z+\frac{y_0y_1^2y_2}{z}\right),\qquad \text{where } U_k=\frac{1}{k}\,\text{Tr}(M^k)\,. \tag{3.5}$$

The last term in the expression of v is necessary because $\text{Tr}(M^6)$ appears to depend explicitly on the spectral parameter z and in order to remove the z dependence of v one needs to have this extra term.

It is important to note that since the Seiberg-Witten curve is based on the dual algebra, to compare our results with the field theory results one should perform a duality transformation as (2.4). Therefore we will consider the theory with the following tree level superpotential

$$W_{\text{tree}}=g_2u-g_6v+\frac{4}{27}g_6u^3\,. \tag{3.6}$$

To find the supersymmetric vacua and the corresponding quantum superpotential we will need to consider the factorization of the Seiberg-Witten curve. More precisely to have an $\mathcal{N}=1$ vacuum there must be some points on the quantum moduli space where monopoles become massless, and at such points the corresponding Seiberg-Witten curve becomes degenerate. The degeneration is such that the Seiberg-Witten curve (3.3) acquires two double roots and two single roots. Having the locus of these singularities one can read the quantum corrected moduli parameters, u and v and thereby find the exact superpotential. To have such a factorization we should impose the following conditions

$$\left.\frac{\partial\mathcal{P}_{\text{quan}}(z,x)}{\partial z}\right|_{z_0,x_0}=0\,,\qquad \mathcal{P}_{\text{quan}}(z_0,x_0)=0\,,\qquad \left.\frac{\partial\mathcal{P}_{\text{quan}}(z,x)}{\partial x}\right|_{z_0,x_0}=0\,. \tag{3.7}$$

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From the first condition one finds $z_0 = \pm\Lambda^4/6$. And therefore the other conditions read

$$\begin{aligned} x_0^6 - 2ux_0^4 + u^2x_0^2 - v \mp 2\Lambda^4\left(x_0^2 - \frac{1}{3}u\right) &= 0\,, \\ 3x_0^4 - 4ux_0^2 + u^2 \mp 2\Lambda^4 &= 0\,. \end{aligned} \tag{3.8}$$

Now the task is to minimize the superpotential subject to the above conditions. A standard procedure is to introduce Lagrange multipliers

$$\begin{aligned} W &= g_2u - g_6v + \frac{4}{27}g_6u^3 + A\left(3x_0^4 - 4ux_0^2 + u^2 \mp 2\Lambda^4\right) + \\ &+ B\left(x_0^6 - 2ux_0^4 + u^2x_0^2 - v \mp 2\Lambda^4\left(x_0^2 - \frac{1}{3}u\right)\right). \end{aligned} \tag{3.9}$$

From the equation of motion for x_0 one finds $A = 0$, while from the equation of motion for v one gets $B = -g_6$. Finally the equation of motion for u leads to the following condition

$$3x_0^3 - 3ux_0^2 + \frac{4}{3}u^2 \mp \Lambda^4 + \frac{3g_2}{2g_6} = 0\,, \tag{3.10}$$

which together with (3.8) can be used to find u, v and x_0 as the following

$$u = 3e^2 \mp \frac{\Lambda^4}{2e^2}\,, \qquad v = \mp 4e^2\Lambda^4 + \frac{\Lambda^8}{3e^2} \mp \frac{\Lambda^{12}}{54e^6}\,, \qquad x_0 = 3e^2 \mp \frac{\Lambda^4}{6e^2}\,, \tag{3.11}$$

where $e = (-g_2/4g_6)^{1/4}$. Therefore the curve is factorized as follows

$$x^6 - 2ux^4 + u^2x^2 - v \mp 2\Lambda^4\left(x^2 - \frac{1}{3}u\right) = \left(x^2 - 3e^2 \pm \frac{\Lambda^4}{6e^2}\right)^2\left(x^2 \pm \frac{2\Lambda^4}{3e^2}\right). \tag{3.12}$$

Setting $\Lambda = 0$ one recognizes this solution as the case where the gauge group is classically broken into $\mathrm{SU}(2) \times \mathrm{U}(1)$. Moreover the quantum superpotential can be obtained from the expression of u and v which is given by

$$\begin{aligned} W &= g_2u + g_6\left(-v + \frac{4}{27}u^3\right) \\ &= 3g_2e^2 + 4g_6e^6 \pm 2\sqrt{-g_2g_6}\Lambda^4\,, \end{aligned} \tag{3.13}$$

in agreement with the field theory result [30].

To study the effective superpotential for the case where classically the gauge group is not broken, we should look for a factorization of the Seiberg-Witten curve in such a way that in the $\Lambda \to 0$ the curve behaves as $\mathcal{P} \sim x^6$. To find such a solution we note that the most general factorization would be as follows

$$x^6 - 2ux^4 + u^2x^2 - v \mp 2\Lambda^4\left(x^2 - \frac{1}{3}u\right) = \left(x^2 - x_\mp^2\right)^2\left(x^2 - y_\mp^2\right), \tag{3.14}$$

which leads to the following conditions

$$\begin{aligned} x_-^6 - 2ux_-^4 + u^2x_-^2 - v - 2\Lambda^4\left(x_-^2 - \frac{1}{3}u\right) &= 0\,, \\ x_+^6 - 2ux_+^4 + u^2x_+^2 - v + 2\Lambda^4\left(x_+^2 - \frac{1}{3}u\right) &= 0\,, \\ 3x_-^4 - 4ux_-^2 + u^2 - 2\Lambda^4 &= 0\,, \\ 3x_+^4 - 4ux_+^2 + u^2 + 2\Lambda^4 &= 0\,, \end{aligned} \tag{3.15}$$

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which can be solved for $x_\pm, u$ and v. The result is

$$u = 3^{1/4} 2\Lambda^2, \qquad v = \frac{4}{3^{1/4}}\,\Lambda^6\,, \qquad x_\mp^2 = \frac{\sqrt{3}\mp 1}{3^{1/4}}\Lambda^2\,. \tag{3.16}$$

The curve is then factorized as

$$x^6 - 2ux^4 + u^2x^2 - v \mp 2\Lambda^4\left(x^2 - \frac{1}{3}u\right) = \left(x^2 - \frac{\sqrt{3}\mp 1}{3^{1/4}}\Lambda^2\right)^2 \left(x^2 - \frac{2(\sqrt{3}\pm 1)}{3^{1/4}}\Lambda^2\right). \tag{3.17}$$

It is now obvious to see that this solution corresponds to the situation where classically the gauge group remains unbroken.

It is also interesting to note that the whole Seiberg-Witten curve is also factorized in this case as follows

$$(3\pm 2\sqrt{3})\left(z + \frac{\Lambda^8}{36z}\right) + x^4 - \frac{2(3\pm\sqrt{3})}{3^{3/4}}\Lambda^2 x^2 \pm \frac{3\pm 2\sqrt{3}}{3}\Lambda^4 = 0\,, \tag{3.18}$$

which should capture the information of the low energy theory.

Finally the effective superpotential reads

$$\begin{aligned} W &= g_2 u - g_6 v + \frac{4}{27} g_6 u^3 \\ &= 4\left(\frac{3^{1/4}}{2} g_2\Lambda^2 - \frac{3^{3/4}}{27} g_6 \Lambda^6\right). \end{aligned} \tag{3.19}$$

We note that in comparison with the case where the gauge group is classically broken to SU(2) $\times$ U(1) the four conditions (3.15) must be satisfied simultaneously, which means that in this case four monopoles become massless. To see this manifestly we note that there are two inequivalent moduli spaces for G_2.[4] In fact the condition $\partial\mathcal{P}_{\rm quan}/\partial z = 0$ besides the solution we have been considering so far, $z_0 = \pm\Lambda^4/6$, has another solution which is given by the following algebraic equation

$$\left(z + \frac{\Lambda^8}{36z}\right) = x^4 - \frac{u}{3}x^2\,, \tag{3.20}$$

which generates a polynomial of eighth order

$$P_8 = 12x^8 - 12ux^6 + 4u^2x^4 - 3vx^2 + \Lambda^8\,. \tag{3.21}$$

Therefore the conditions for having a degenerate curve are now given by

$$P_8|_{x_0} = 0\,, \qquad \left.\frac{\partial P_8}{\partial x}\right|_{x_0} = 0\,. \tag{3.22}$$

One can now proceed to find the points on the moduli space where the curve becomes degenerate. Doing so, we will get the same solution as before, namely for the case where the gauge group is classically broken to SU(2) $\times$ U(1) one finds

$$u = 3e^2 \mp \frac{\Lambda^4}{2e^2}\,, \qquad v = \mp 4e^2\Lambda^4 + \frac{\Lambda^8}{3e^2} \mp \frac{\Lambda^{12}}{54e^6}\,, \tag{3.23}$$

[4]Having two copies of the moduli space for G_2 has also been noticed in [30].

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and the curve is factorized as

$$12x^8 - 12ux^6 + 4u^2x^4 - 3vx^2 + \Lambda^8 = 12\left(x^2 \pm \frac{\Lambda^4}{6e^2}\right)^2\left(x^4 - 3x^2e^2 + 3e^4 \pm x^2\frac{\Lambda^4}{e^2}\right). \quad (3.24)$$

On the other hand for the situation where classically the gauge group remains unbroken the solution for the parameters of the moduli space is the same as before, i.e. $u = 3^{1/4}2\Lambda^2$, $v = \frac{4}{3^{1/4}}\Lambda^6$, and the curve is factorized as

$$12x^8 - 12ux^6 + 4u^2x^4 - 3vx^2 + \Lambda^8 = 12\left(x^2 - \frac{\sqrt{3}+1}{2\times 3^{1/4}}\Lambda^2\right)^2\left(x^2 - \frac{\sqrt{3}-1}{2\times 3^{1/4}}\Lambda^2\right)^2. \quad (3.25)$$

As we see these two solutions give two different factorizations in this branch. Indeed in the first case we get the points where two monopoles become massless while in the second one we get the points where four monoploes become massless.

One could also consider the points of the moduli space where some mutually non-local monopoles become massless. These would lead to a superconformal field theory [34, 35]. In the G_2 case from the first branch where we get a polynomial of sixth order, there is only one way to get such a point where the curve is factorized as

$$x^6 - 2ux^4 + u^2x^2 - v \mp 2\Lambda^4\left(x^2 - \frac{1}{3}u\right) = (x^2 - x_0^2)^3\,. \quad (3.26)$$

which means that one should impose the condition that the curve has triplet roots. Doing so one gets the following solution for u and v

$$u = \sqrt{6}\Lambda^2\,, \qquad v = \frac{10\sqrt{6}}{9}\Lambda^6\,, \quad (3.27)$$

and the curve is factorized only for plus sign as follows

$$x^6 - 2ux^4 + u^2x^2 - v + 2\Lambda^4\left(x^2 - \frac{1}{3}u\right) = \left(x^2 - \frac{2\sqrt{6}}{3}\Lambda^2\right)^3. \quad (3.28)$$

Doing same for the second branch one finds

$$12x^8 - 12ux^6 + 4u^2x^4 - 3vx^2 + \Lambda^8 = 12\left(x^2 - \frac{\sqrt{6}}{2}\Lambda^2\right)\left(x^2 - \frac{\sqrt{6}}{6}\Lambda^2\right)^3. \quad (3.29)$$

This case corresponds to the situation where the discriminant of the quantum curve and its derivative with respect to u and v vanish [30].

4. Theory on $R^3 \times S^1$

Let us now consider the $\mathcal{N} = 1$ G_2 supersymmetric gauge theory on $R^3 \times S^1$ space. This model can be thought of as an $\mathcal{N} = 2$ SYM theory on $R^3 \times S^1$ deformed by a tree level potential $\mathrm{Tr}\, W(\phi)$. To study the quantum theory one may use the corresponding integrable model. Actually since the moduli of the quantum curve parameterize the quantum moduli

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space of the theory, one might suspect that $\mathrm{Tr}(M^k)$ is the quantum corrected version of the classical gauge invariant parameter $\mathrm{Tr}(\phi^k)$.[5] In other words, in view of the proposal made in [15] the effective superpotential should be $\mathrm{Tr}\, W(M)$. To be specific, consider a deformation of the theory with the following tree level superpotential

$$W = g_2 u + g_6 \left(-v + \frac{4}{27}u^3\right) = \frac{1}{4}g_2\,\mathrm{Tr}(\phi^2) - \frac{1}{6}g_6\,\mathrm{Tr}(\phi^6) + \frac{11}{864}g_6\,\mathrm{Tr}(\phi^2)^3\,. \tag{4.1}$$

According to the proposal of [15] the effective superpotential is given by

$$\begin{aligned} W = & \frac{1}{4}g_2\,\mathrm{Tr}(M^2) - \frac{1}{6}g_6\,\mathrm{Tr}(M^6) + \frac{11}{864}g_6\,\mathrm{Tr}(M^2)^3 - \frac{5}{2}g_6\,\mathrm{Tr}(M^2)\left(z + \frac{y_0y_1^2y_2}{z}\right) + \\ & + L\log\left(\frac{y_0y_1^2y_2}{\Lambda^8/36}\right). \end{aligned} \tag{4.2}$$

Here we have also imposed the constraint on y_i using a Lagrange multiplier L.

4.1 Case 1: unbroken gauge symmetry

To get a supersymmetric vacuum one needs to minimize this superpotential with respect to ϕ_i and y_i. This can be done using the equations of motion of the fields ϕ_i and y_i. The corresponding equations are given in appendix A. One can easily see that one solution to the equations is given by

$$\phi_1 = \phi_2 = 0\,, \qquad y_1 = \frac{2}{3}y\,, \qquad y_2 = \frac{1}{3}y\,, \qquad y_0 = y\,, \tag{4.3}$$

where $y = 3^{1/4}\Lambda^2/2$. This solution corresponds to the situation in which the gauge group is classically unbroken. Moreover the gauge invariant parameters read $u = 3^{1/4}2\Lambda^2$ and $v = 4/3^{1/4}\Lambda^6$. Plugging this solution into the expression of the effective potential one finds

$$W = 4\left(\frac{3^{1/4}}{2}g_2\Lambda^2 - \frac{3^{3/4}}{27}g_6\Lambda^6\right), \tag{4.4}$$

in agreement with (3.19).

This result can be used to study the four dimensional theory on R^4. For example let us use this result to integrate in the glueball field S for the theory on R^4. To do this we note that the Lagrange multiplier can be interpreted as the glueball field. To integrate it in one needs to minimize the effective superpotential without using its equation of motion for L, and we get (replacing L by S)

$$W = 4g_2y - \frac{32}{27}g_6y^3 + S\log\left(\frac{3\Lambda^8}{16y^4}\right). \tag{4.5}$$

The next step is to integrate out y

$$\frac{\partial W}{\partial y} = 4g_2 - \frac{32}{9}g_6y^2 - 4\frac{S}{y} = 0\,, \tag{4.6}$$

[5] Actually, the precise statement is that the z-independent part of $\mathrm{Tr}(M^k)$ is the quantum corrected version of $\mathrm{Tr}(\phi^k)$.

which is an equation that can be used to solve for y. In fact the solution can be given in power series of S. Up to $\mathcal{O}(S^8)$ one finds

$$y = \frac{1}{g_2}S + \frac{8}{9}\frac{g_6}{g_2^4}S^3 + \frac{64}{27}\frac{g_6^2}{g_2^7}S^5 + \frac{2048}{243}\frac{g_6^3}{g_2^{10}}S^7\,. \tag{4.7}$$

Plugging the above expression for y into the effective superpotential one gets

$$W = -4S\left(\log\left(\frac{2S}{3^{1/4}g_2\Lambda^2}\right) - 1\right) - \frac{32}{27}\frac{g_6}{g_2^3}S^3 - \frac{128}{81}\frac{g_6^2}{g_2^6}S^5 - \frac{8192}{2187}\frac{g_6^3}{g_2^9}S^7\,. \tag{4.8}$$

up to order eight in the glueball field S.

4.2 Case 2: G_2 broken to $\mathrm{SU}(2)\times\mathrm{U}(1)$

The equations of motion coming from the potential (4.2) also have another solution. In fact it can also be seen that the following ansatz solves the equations given in appendix A,

$$\phi_1 = \phi\,, \qquad \phi_2 = 2\phi\,, \qquad y_0 = \pm\frac{4\Lambda^4}{9e^2}\,, \qquad y_1 = -\frac{3e^2}{4}\,, \qquad y_2 = \pm\frac{\Lambda^2}{9e^2}\,, \tag{4.9}$$

where $e = (-g_2/4g_4)^{(1/4)}$ and ϕ is given by

$$\phi^2 = -\frac{1}{12}\left(e^2 \pm \frac{10\Lambda^4}{27e^2}\right). \tag{4.10}$$

Looking at the limit $\Lambda \to 0$, one can see that this solution corresponds to the situation where the gauge group is classically broken into $\mathrm{SU}(2)\times\mathrm{U}(1)$ (see (2.9)).

The gauge invariant parameters are found to be

$$u = 3e^2 \mp \frac{\Lambda^4}{2e^2}\,, \qquad v = \mp 4e^2\Lambda^4 + \frac{\Lambda^8}{3e^2} \mp \frac{\Lambda^{12}}{54e^6}\,, \tag{4.11}$$

and therefore the quantum superpotential reads

$$W = 3g_2e^2 + 4g_6e^6 \pm 2\sqrt{-g_2g_6}\Lambda^4\,, \tag{4.12}$$

which is the same as the one we found in the field theory.

Since quantum mechanically the gauge symmetry is broken to U(1), we expect a free parameter in the solution. Note that the situation differs from the $\mathrm{U}(N)$ case, there we had a center of mass U(1) that did not manifest itself in the solution. Because G_2 is a simple Lie group we do not have this center of mass U(1). Thus we would expect to see one free parameter in our solution. Obviously the solution (4.9) does not have a free parameter, which means that the ansatz $\phi_2 = 2\phi_1 = 2\phi$ somehow fixes this parameter. So the solution we have found is merely a special case in a one-parameter family of solutions.

In fact the situation is very similar to the $\mathcal{N} = 1$ SU(3) case where the gauge group is classically broken into $\mathrm{SU}(2)\times\mathrm{U}(1)$. Similarly to the G_2 model, in the IR limit the SU(2) factor of remaining gauge group gets confined and we are left with only U(1). Therefore one would expect to get a one-parameter family of solutions. To see this let us consider

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the $\mathcal{N} = 1$ SU(3) case on $R^3 \times S^1$ in more detail (see also [15]). Consider the tree level superpotential

$$W = \frac{g_2}{2}\,\mathrm{Tr}(\phi^2) + \frac{g_3}{3}\,\mathrm{Tr}(\phi^3)\,, \tag{4.13}$$

where ϕ is the adjoint scalar. The quantum superpotential is then given by

$$W = \frac{g_2}{2}\,\mathrm{Tr}(M^2) + \frac{g_3}{3}\,\mathrm{Tr}(M^3)\,, \tag{4.14}$$

with

$$M = \begin{pmatrix} \phi_1 & y_1 & z \\ 1 & \phi_2 & y_2 \\ \frac{y_0}{z} & 1 & -\phi_1-\phi_2 \end{pmatrix} \tag{4.15}$$

being the Lax matrix of the corresponding integrable model.

One can write down the equations for the extrema of W and then solve it. For the situation we are interested in, where the gauge group is classically broken into $\mathrm{SU}(2)\times\mathrm{U}(1)$ one could start from an ansatz in which $\phi_1 = \phi$, $\phi_2 = -2\phi$ and solve the equation. Doing so one finds

$$\phi = \frac{g_2}{g_3} \pm \frac{\Lambda^2}{\phi_0}\,, \qquad y_0 = \frac{\Lambda^4}{\phi_0^2}\,, \qquad y_1 = \phi_0\Lambda\,, \qquad y_2 = \phi_0\Lambda\,, \tag{4.16}$$

where ϕ_0 is a solution of the following equation

$$g_3\phi_0^3 \pm 3g_2\Lambda\phi_o + 2g_3\Lambda^3 = 0\,. \tag{4.17}$$

On the other hand relaxing the condition $\phi_2 = -2\phi_1$ one can also find other solutions, namely

$$\phi_1 = \frac{g_2}{g_3} \pm \frac{\Lambda^3}{y}\,, \qquad \phi_2 = \frac{-2g_2}{g_3} \mp \frac{\Lambda^3}{y} \mp \frac{1}{\phi_0}\,, \qquad y_0 = \frac{\Lambda^3}{y\phi_0}\,, \qquad y_1 = \phi_0\Lambda^3\,, \qquad y_2 = y, \tag{4.18}$$

where ϕ_0 is a solution of the following equation

$$g_3y^2\phi_0^2 + (3g_2y \pm g_3\Lambda^3)\phi_0 + g_3y = 0\,, \tag{4.19}$$

which is a one-parameter solution, as expected. Therefore the solution with $\phi_2 = -2\phi_1$ is merely a special case in a one-parameter family of solutions. Nevertheless we note, however, that both of these solutions lead to the same superpotential which is

$$W = \frac{g_2^3}{g_3^2} \pm 2g_3\Lambda^3\,. \tag{4.20}$$

Obviously, the free parameter corresponds to a flat direction, and as far as the superpotential is concerned having one special solution is enough to determine the value of the superpotential in the supersymmetric vacua.

Nevertheless we can still see the existence of the free parameter by considering flows in the integrable system whose Lax matrix we are using in constructing the quantum superpotential. In general there are two independent flows, generated by M^2 and M^6, or

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equivalently by u and v. The flows of the integrable system act on the Lax matrix via commutators

$$\frac{\partial}{\partial t_k} M = [M, \mathcal{L}(M^{k-1})] , \tag{4.21}$$

but one can also consider the flows of the dynamical variables $\xi \in \{\phi_i, y_j\}$ by calculating the Poisson brackets

$$F_k(\xi) = \frac{\partial}{\partial t_k} \xi = \{\xi, \operatorname{Tr} M^k\} . \tag{4.22}$$

To calculate the Poisson brackets one has to identify the coordinates that correspond to the conjugate momenta ϕ_i, these are the x's appearing in

$$y_i = \exp(\alpha_i \cdot x) , \tag{4.23}$$

with the α_i the simple roots of $D_4^{(3)}$

$$\alpha_0 = -(2\alpha_1 + \alpha_2) , \qquad \alpha_1 = (0, \sqrt{2}) , \qquad \alpha_2 = \left(\frac{1}{2}\sqrt{6}, -\frac{3}{2}\sqrt{2}\right) . \tag{4.24}$$

The Poisson brackets then read

$$\{\phi_i, y_j\} = (\alpha_j)_i y_j , \tag{4.25}$$

where $(\alpha_j)_i$ is the i-th component of the j-th root in the basis (4.24). Using these brackets it is straightforward to calculate the flows $F_k(\xi)$. We find that for all $\xi \in \{\phi_i, y_j\}$ the two flows F_2 and F_6 are related in the following way:

$$F_6(\xi) = -\frac{9}{4}\left(3\frac{m}{g} + 176\Lambda^4 + 2112\Lambda^8 \frac{g}{m}\right) F_2(\xi) . \tag{4.26}$$

So, indeed, there is exactly one independent flow and therefore precisely one free parameter in our solution. This establishes the fact that the symmetry is broken down to a single U(1).

5. Factorization

5.1 Deriving the resolvent for G_2

In this section we present some preliminary results that are a first step towards generalizing the algebraic geometric proof of the factorization of Seiberg-Witten curves given in section 4 of [17]. The results will be applied to G_2. A more detailed treatment will be given in a future publication.

It is a well-known fact that the Seiberg-Witten curve has an underlying integrable system. The integrable system is characterized by the existence of a complete set of action-angle variables. In terms of these variables the evolution in the phase-space of the classical mechanical system becomes quite simple, half of the variables are conserved and the other half (the angle variables) evolve with constant velocity. Further, to the integrable system one can associate a Riemann surface, which is equivalent to the Seiberg-Witten curve. The conserved quantities then correspond to the moduli of this surface and the angle variables are coordinates on the jacobian of this Riemann surface.[6] The equations of motion of the integrable system correspond to linear flows on the jacobian.

The main idea of section 4 of [17] is that the superpotential is at an extremum if the velocities of the flows on the jacobian are zero. The velocities of the flows are expressed in terms of the superpotential $W(x)$ and the one-forms ω_k (see [36]):

$$v_k(W) = \mathrm{res}_{x=\infty}\left(W'(x)\omega_k\right) . \tag{5.1}$$

Let us also remind the reader that we can express the quantum superpotential as a residue

$$W_{\text{quantum}} = \mathrm{res}_{x=\infty}\left(W(x)R(x)\right) ,$$

with $R(x)$ the gauge theory resolvent,

$$R(x) = \mathrm{Tr}\,\frac{1}{x-\Phi} . \tag{5.2}$$

It turns out to be possible to express both the one-forms ω_k and the resolvent in terms of a single function:

$$\Omega(x, u_k) = \left(\log\det(x - M(z))\right)|_{z^0} , \tag{5.3}$$

by which we mean the z-independent part of $\log\det(x - M(z))$. Further, $M(z)$ is the Lax matrix (with spectral parameter z) of the integrable system that underlies the Seiberg-Witten curve and the u_k are the moduli of this curve. As it stands, Ω is not well defined, because we have to extract the z-independent part of some complicated function with branch cuts. One way to define Ω is as follows. Since the characteristic polynomial $\det(x - M(z))$ is symmetric under the interchange of z and $1/z$, we can write

$$\det(x - M(z)) = a_0^2 \prod_{t=1}^{r}(a_t - z)\left(a_t - \frac{1}{z}\right)$$

and this allows us to define

$$\Omega \equiv 2\sum_{t=0}^{r}\log a_t .$$

Having defined Ω, the resolvent is given by

$$R(x) = \partial_x \Omega(x, u_k) \tag{5.4}$$

and the one-forms by

$$\omega_k = \frac{\partial\Omega}{\partial u_k}\mathrm{dx} . \tag{5.5}$$

Actually, the definition of Ω still suffers from minus sign ambiguities, which will be fixed by demanding that the resolvent that follows from this Ω has the expansion

$$R(x) = \sum_{i=1}^{\infty}\frac{\mathrm{tr}(M(z)^{i-1})|_{z^0}}{x^i} . \tag{5.6}$$

[6] In general, only a subset of the moduli correspond to action variables, and the number of flows need therefore not be equal to the dimension of the jacobian.

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In order to show that this proposal makes sense, we will calculate the resolvent and one-forms for $\mathrm{U}(N)$. From the curve for $\mathrm{U}(N)$

$$\det(x - M(z)) = P_N(x) + (-1)^N \left(z + \frac{1}{z}\right)$$

we easily derive that $a_0^2 = 1/a_1$ and

$$a_1 = \frac{(P + \sqrt{P^2 - 4})}{2}.$$

This yields for the function Ω

$$\Omega = \log\left(\frac{(P + \sqrt{P^2 - 4})}{2}\right),$$

from which we derive the usual resolvent

$$R(x) = \partial_x \Omega = \frac{P'(x)}{\sqrt{P(x)^2 - 4}} \tag{5.7}$$

and one-forms

$$\omega_k = \partial_{u_k} \Omega \mathrm{dx} = \frac{x^{N-k}}{\sqrt{P(x)^2 - 4}} \mathrm{dx}. \tag{5.8}$$

In order to apply this procedure to G_2 we must first compute the roots a_t for the G_2 curve. The algebraic curve is given by

$$3\left(z - \frac{\Lambda^8}{36z}\right)^2 - x^2\left(z + \frac{\Lambda^8}{36z}\right)(6x^2 - 2u) - x^2 P(x) = 0 \tag{5.9}$$

written in terms of $y = z + \Lambda^8/36z$ this reads

$$3y^2 - x^2 y(6x^2 - 2u) - x^2 P(x) - \frac{\Lambda^8}{3} = 0. \tag{5.10}$$

This equation has two solutions:

$$y_\pm = x^2\left(x^2 - \frac{u}{3}\right) \pm \frac{1}{3}\sqrt{x^4(3x^2 - u)^2 + 3x^2 P(x) + \Lambda^8}, \tag{5.11}$$

yielding the four roots of the algebraic curve

$$z_{+\pm} = \frac{1}{2}y_+ \pm \frac{1}{6}\sqrt{9y_+^2 - \Lambda^8}, \qquad z_{-\pm} = \frac{1}{2}y_- \pm \frac{1}{6}\sqrt{9y_-^2 - \Lambda^8}. \tag{5.12}$$

To write down Ω we have to make a choice for the roots. One should pick one root from $\{z_{++}, z_{+-}\}$ and one from $\{z_{--}, z_{-+}\}$, so there are four possible choices:

$$\begin{aligned}\Omega &= \eta(\log z_{++} + \epsilon \log z_{--}), \qquad \text{with } \epsilon^2 = \eta^2 = 1 \\ &= \eta\left(\log\left(\frac{1}{2}y_+ + \frac{1}{6}\sqrt{9y_+^2 - \Lambda^8}\right) + \epsilon \log\left(\frac{1}{2}y_- + \frac{1}{6}\sqrt{9y_-^2 - \Lambda^8}\right)\right).\end{aligned} \tag{5.13}$$

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If we choose $\eta = 1, \epsilon = -1$ the resolvent reads

$$R(x) = \partial_x \Omega = \frac{3\partial_x y_+}{\sqrt{9y_+^2 - \Lambda^8}} - \frac{3\partial_x y_-}{\sqrt{9y_-^2 - \Lambda^8}}\,. \tag{5.14}$$

The expansion of the resolvent around $x = \infty$ should have the form of (5.6). Indeed, when we do the expansion we get

$$\begin{aligned} R(x) &= \frac{8}{x} + \frac{4u}{x^3} + \frac{4u^2}{x^5} + \frac{4u^3 + 6v}{x^7} + \frac{4u^4 + 16uv + \frac{20\Lambda^8}{3}}{x^9} + \\ &+ \frac{4u^5 + 30u^2v + 30u\Lambda^8}{x^{11}} + \frac{4u^6 + 48u^3v + 6v^2 + \frac{250}{3}\Lambda^8}{x^{13}} + \mathcal{O}\left(\frac{1}{x^{15}}\right). \end{aligned} \tag{5.15}$$

One can check that the coefficients in this expansion correspond to the traces of powers of the Lax matrix. Classically, one could write for the resolvent: $P_6'(x)/P_6(x)$, this would generate the correct expansion if one would set to zero the terms in the expansion that explicitly depend on Λ. Apparently this naive guess for the resolvent is correct up to order $1/x^7$.

This resolvent also hints at the existence of a hyper-elliptic curve for G_2. This can be seen as follows. The resolvent can be written in the form

$$R(x) = \frac{r(x)}{x^2\sqrt{P_6(x)^2 - 4\Lambda^8(x^2 - u/3)^2}} \tag{5.16}$$

with $r(x)$ some function without poles. Comparing this resolvent to that of $\mathrm{U}(N)$ leads us to suggest that

$$y^2 = P_6(x)^2 - 4\Lambda^8\left(x^2 - \frac{u}{3}\right)^2 \tag{5.17}$$

is in fact a hyper-elliptic curve for G_2. Indeed, in analyzing the factorization of the G_2 curve, expressions like $P_6(x) \pm 2\Lambda^4(x^2 - u/3)$ pop up everywhere.

Notice that the resolvent of the gauge theory contains arbitrarily high powers of the adjoint scalar field. The precise definition of such operators in the quantum theory depends on a choice of UV completion of the theory. The integrable system prefers one particular UV completion, which is the one where we define

$$\mathrm{tr}(\Phi^i) \equiv \mathrm{tr}(M(z)^i)|_{z^0}\,. \tag{5.18}$$

In the case of $\mathrm{U}(N)$, this was also the UV completion preferred by string theory. We see that the integrable system provides a natural UV completion for the exceptional gauge groups as well. It would be interesting to explore other UV completions, e.g. those obtained by taking the Lax matrix in another representation, but we leave that for future work.

We now want to use the flow equations (5.1) to determine the minima of the superpotential. One therefore has to calculate the one-forms $\partial_{u,v}\Omega$

$$\omega_u = \partial_u\Omega = \partial_u a(x)R_1(x) + \partial_u b(x)R_2(x) \tag{5.19}$$

$$\omega_v = \partial_v\Omega = \partial_v a(x)R_1(x) + \partial_v b(x)R_2(x)\,. \tag{5.20}$$

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The conditions that the flows on the jacobian vanish ($v_l = 0$) then imply:

$$\begin{aligned} x^2 R_2(x) W'(x) &= r_v(x) + \sum_{l=1} \frac{c_l}{x^{2l+1}} \\ \left(-\frac{x^2}{3} R_1(x) + 4x^4(2u-3x^2)R_2(x)\right) W'(x) &= r_u(x) + \sum_{l=1} \frac{d_l}{x^{2l+1}} \,. \end{aligned} \tag{5.21}$$

The flow equations for the U(N) case allowed us to derive the factorization of the gauge theory and Matrix model curve (see [17] section 4), in a similar spirit equations (5.21) should somehow define the analogue of the Matrix model curve for G_2. Unfortunately, we have not yet succeeded in writing (5.21) in a more manageable form, and it is therefore harder to draw general conclusions from these equations. In the next section we will use an alternative method to work out the factorization of the Matrix model curve for a superpotential with terms up to order six.

In the remaining part of this section we will show that (5.21) is indeed equivalent to minimizing the superpotential. For definiteness we will choose the superpotential to be $W'(x) = g_2 x + g_6 x^5$. The values of u and v in the minimum determine the Seiberg-Witten curve completely and therefore also the factorization properties of this curve. To study the factorization properties it is useful to consider the conditions for the curve to develop a double zero:

$$P_6(x_0) = \pm 2\Lambda^4 \left(x_0^2 - \frac{u}{3}\right) \tag{5.22}$$

$$P_6'(x_0) = \pm 4\Lambda^4 x_0 \,, \tag{5.23}$$

these equations can be used to solve for u and v in terms of x_0. So there is only one free parameter, not two. Therefore the two equations (5.21) are replaced by the single equation

$$\mathrm{res}_{x=\infty} \left(\partial_{x_0} \Omega W'(x)\right) = 0. \tag{5.24}$$

This equation can be used to solve for x_0, which will allow us to express u and v in terms of the coupling constants and the energy scale Λ. One can then substitute u and v into equation (5.22) and study its factorization properties. For the superpotential $W'(x) = g_2 x + g_6 x^5$ we find three classes of solutions (note that we consider single trace operators here, so these results should not be compared with the results from the previous sections)

1. $x_0 = 0 \Rightarrow P_6(x) - 2\Lambda^4 \left(x^2 - \frac{u}{3}\right) = x^4(x^2 - 2\sqrt{2}\Lambda^2)$
2. $x_0 = \eta \left(\frac{8}{3}\right)^{1/4}, \qquad \eta^4 = -1 \Rightarrow P_6(x) - 2\Lambda^4 \left(x^2 - \frac{u}{3}\right) = \left(x^2 \pm 2\mathrm{i}\sqrt{\frac{2}{3}}\Lambda^2\right)^3$

This solution is similar to the superconformal solution.

3. $x_0 = \epsilon \left(\Lambda^4 - 6e - \frac{5}{33}\sqrt{\frac{9}{e^2} - 66\frac{\Lambda^4}{e} + 22\Lambda^8}\right)^{1/4}, \qquad \epsilon^4 = 1\,, \qquad e = \frac{g_6}{g_2}$
$\Rightarrow P_6(x) - 2\Lambda^4 \left(x^2 - \frac{u}{3}\right) = (x^2 - \alpha)(x^2 - \beta)^2$

Here α and β are some (messy) expressions in e and Λ.

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In order to check the claim that equation (5.24) is equivalent to minimizing the superpotential, if suffices to minimize

$$W = \frac{g_2}{2}u + \frac{g_6}{6}v + A\left(P_6(x_0) \pm 2\Lambda^4\left(x_0^2 - \frac{u}{3}\right)\right) + B(P_6'(x_0) \pm 4\Lambda^4 x_0) \tag{5.25}$$

with respect to u, v, A, B and x_0. The calculations are pretty straightforward and we find complete agreement, suggesting that Ω indeed generates the one-forms as described.

5.2 Extremization problem: proof of $B_{l-1}^2 F_{12-2l} = W'(x)^2 + f_8(x)$

In order to understand the curve factorization better we apply the same analysis as in [4] to the G_2 case. We consider a single trace superpotential and for the matrix Φ we consider three independent fields ϕ_1, ϕ_2, ϕ_3 as the non zero diagonal components. We know that these three fields are not in fact independent and classically

$$\phi_3 = \phi_2 - \phi_1\,. \tag{5.26}$$

At the quantum level the following constraint holds

$$u_4 = \left(\frac{u_2}{2}\right)^2. \tag{5.27}$$

We impose this constraint by a Lagrange multiplier, C. The effective superpotential will read

$$\begin{aligned} W_{\text{eff}} = & \sum_{r=1}^{3} g_{2r}u_{2r} + \left[L_i\left(P_6(x) - 2\epsilon_i\Lambda^4\left(x^2 - \frac{u_2}{6}\right)\right)\Big|_{x=p_i} + Q_i\left(\frac{\partial}{\partial x}P_6 - 4\epsilon_i\Lambda^4 x\right)\Big|_{x=p_i}\right] + \\ & + C\left(u_4 - \left(\frac{u_2}{2}\right)^2\right), \end{aligned} \tag{5.28}$$

where l is the number of double zeroes, ϵ is a second root of unity and the p_i are the points where the factorization occurs. In lines parallel to [4] one can see that $Q_i = 0$ and

$$P_6 = \langle \det(xI - \Phi)\rangle = \sum_{j=0}^{\infty} x^{6-j} s_l\Big|_+ , \tag{5.29}$$

where "+" means the polynomial part of the series. Using the relation, $\partial s_j/\partial u_k = -s_{j-k}$ and upon variation of (5.28) with respect to all u_r one finds

$$\begin{aligned} g_2 &= \sum_{i=1}^{l} L_i\left[\sum_{j=0}^{6} p_i^{6-j}s_{j-2} - \epsilon_i\frac{\Lambda^4}{3}\right] + C\frac{u_2}{2} \\ g_4 &= \sum_{i=1}^{l}\sum_{j=0}^{6} L_i p_i^{6-j}s_{j-4} - C \\ g_6 &= \sum_{i=1}^{l} L_i \end{aligned} \tag{5.30}$$

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Multiplying (5.28) by x^{2r-1} and summing over r and imposing the L_i constraints one will find

$$
\begin{aligned}
W'(x) &= \sum_{r=1}^{3} g_{2r} x^{2r-1} \\
&= \sum_{r=-\infty}^{3} \sum_{i=1}^{l} \sum_{j=0}^{6} x^{2r-1} p_i^{6-j} s_{j-2r} L_i - \sum_{i=1}^{l} 2L_i \epsilon_i \Lambda^4 x^{-1} \left(p_i^2 - \frac{u_2}{6}\right) + \\
&\quad + C\left(\frac{u_2}{2}x - x^3\right) - L\frac{\Lambda^4}{3}x + \mathcal{O}(x^{-2}) \\
&= \sum_{i=1}^{l} \frac{P_6(x;\langle u\rangle)}{x-p_i} L_i - \sum_{i=1}^{l} 2L_i \epsilon_i \Lambda^4 x^{-1}\left(p_i^2 - \frac{u_2}{6}\right) + C\left(\frac{u_2}{2}x - x^3\right) - \\
&\quad - L\frac{\Lambda^4}{3}x + \mathcal{O}(x^{-2})\,,
\end{aligned} \tag{5.31}
$$

where $L \equiv \sum_{i=1}^{l} L_i \epsilon_i$. Defining $B_{l-1}(x)$ by

$$
\sum_{i=1}^{l} \frac{L_i}{x-p_i} = \frac{B_{l-1}(x)}{H_l(x)} \tag{5.32}
$$

one has

$$
\begin{aligned}
W'(x) + \sum_{i=1}^{l} 2L_i\epsilon_i\Lambda^4 x^{-1}\left(p_i^2 - \frac{u_2}{6}\right) - C\left(\frac{u_2}{2}x - x^3\right) + L\frac{\Lambda^4}{3}x &= \\
= B_{l-1}(x)\sqrt{F_{12-2l}(x) + \frac{4\Lambda^8(x^2 - \frac{u_2}{6})^2}{H_l(x)^2}} + \mathcal{O}(x^{-2})\,. &
\end{aligned} \tag{5.33}
$$

Squaring (5.33) we find

$$
B_{l-1}^2 F_{12-2l} = W'(x)^2 + 2g_6 C x^8 + \mathcal{O}(x^6)\,, \tag{5.34}
$$

which is the desired result. This suggests that the right hand side is somehow related to the matrix model curve and that therefore the appropriate matrix model curve may well be hyperelliptic, just as we found hints that the gauge theory curve may also be represented in hyperelliptic form, as we discuss in the next section.

6. Hyperelliptic curve for G_2

Our considerations in section 5 about the resolvent for G_2 suggest that the exact result for $\mathcal{N}=2$ G_2 SYM theory can be obtained from a hyperelliptic curve given by

$$
y^2 = (x^6 - 2ux^4 + u^2x^2 - v)^2 - 4\Lambda^8\left(x^2 - \frac{u}{3}\right)^2. \tag{6.1}
$$

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Actually having a hyperelliptic curve for G_2 was first suggested in [37] though the proposed curve leads to incorrect singularities of the moduli space [30]. Therefore it was believed that the correct curve for G_2 which comes from the integrable model need not be hyperelliptic. The corresponding hyperelliptic curve is [37]

$$y^2 = (x^6 - 2ux^4 + u^2x^2 - v)^2 - 4\Lambda^8 x^4 \,. \tag{6.2}$$

As we see the hyperelliptic curve (6.1) suggested by the resolvent of G_2 is a simple modification of this curve, though this simple modification leads to completely different physics.

To study the singularity structure of the hyperelliptic curve let us consider the case where the gauge group is classically broken to $\mathrm{SU}(2)\times\mathrm{U}(1)$. From the field theory considerations we know that the quantum corrections to the gauge invariant variables are given by (3.11). Upon eliminating e one finds

$$\Delta_{\mp}^{\text{field th.}} = \pm 12u^3v \mp 81v^2 - 108vu\Lambda^4 + 16\Lambda^4u^4 \pm 12u^2\Lambda^8 + 96\Lambda^{12} = 0\,, \tag{6.3}$$

as the condition for a vacuum with a massless dyon. Note that $\Delta_{+}^{\text{field th.}}$ and $\Delta_{-}^{\text{field th.}}$ intersect transversally in four points (see equation (3.16))

$$(u,v) = \left(e^{\frac{in\pi}{2}}3^{1/4}2\Lambda^2, -e^{\frac{3in\pi}{2}}\frac{4}{3^{1/4}}\Lambda^6\right), \qquad \text{for } n = 0,1,2,3\,, \tag{6.4}$$

which is equal to number of the supersymmetric ground states of $\mathcal{N}=1$ G_2 SYM theory.

Let us now consider the hyperelliptic curve (6.2). The discriminant of the curve is given by

$$\Delta^h = v\Delta_+^h\Delta_-^h \tag{6.5}$$

with

$$\Delta_{\pm}^h = 27v^2 - 4vu^3 \mp 72uv\Lambda^4 \pm 8u^4\Lambda^4 + 32u^2\Lambda^8 \pm 32\Lambda^{12}\,. \tag{6.6}$$

It can be shown [30] that this leads to an incorrect number of $\mathcal{N}=1$ vacua and moreover the overall factor of v in the discriminant gives a monodromy which is not present in the Weyl group of G_2. Therefore one might conclude that the hyperelliptic curve (6.2) is not a proper curve describing $\mathcal{N}=2$ G_2 SYM theory.

On the other hand the discriminant of the hyperelliptic curve (6.1) is given by

$$\Delta = (4u^2\Lambda^8 - 9v^2)(-4u^3 + 27v)\Delta_{+}^{\text{field th.}}\Delta_{-}^{\text{field th.}}\,. \tag{6.7}$$

In comparison with the field theory result the discriminant has two extra overall factors. The first one was already present in the G_2 curve coming from the integrable system (3.3) which is believed to be an accidental singularity [30]. The second one, $-4u^3+27v$, is present in our hyperelliptic curve. Nevertheless since $\Delta_{\pm}^{\text{field th.}}$ in the discriminant of the curve (6.1) coincide precisely with the gauge theory condition for having massless dyons, (6.3), one might suspect that this singularity is accidental too. Moreover, by construction, the curve

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also gives the correct factorization. Therefore one could believe that the curve (6.1) is a proper curve describing $\mathcal{N}=2$ G_2 SYM theory. Definitely this issue deserves to be studied more carefully.

7. Conclusion

We have studied $\mathcal{N}=2$ G_2 SYM theory deformed by a tree level superpotential on $R^3 \times S^1$ using the corresponding integrable model of the theory which is the periodic Toda lattice based on the dual affine G_2. For the cases where the gauge group is classically broken and where it remains unbroken we have obtained the vacuum structure and the exact superpotential of the theory both by conventional field theory methods (Seiberg-Witten curve factorization) and by integrable model techniques (using the Lax operator) and we have shown complete agreement between the two approaches in each case.

We have also put forward a general recipe for deriving the resolvent and the one-forms from the Seiberg-Witten curve. This method was applied to the Seiberg-Witten curve for G_2. Using the one-forms to calculate the flows on the jacobian we reproduced the conditions for an extremum of the superpotential, as the flow equations should. Also, the proposed resolvent appears to be correct, since it reproduces the correct expansion around $x=\infty$. The resolvent obtained from the Lax matrix provides a natural UV completion of the theory, and we also found a set of algebraic equations that somehow encode the appropriate notion of a matrix model curve for the gauge group G_2. Clearly, more work is needed to determine the precise structure of this generalized matrix model curve.

The extremization problem has also been considered with a proof allowing us to state the problem in purely algebraic terms. Contradicting earlier beliefs we have presented evidences and arguments supporting the existence of a hyperelliptic curve for G_2. This last suggestion deserves further study which we postpone to future work.

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A. Equations of motion

Here we present the equations of motion for the potential given in (4.2). For ϕ_1 we get

$$
\begin{aligned}
0 = 3g_2(2\phi_1-\phi_2)+2g_6\Big(&-27y_2y_1\phi_2+9y_1y_0\phi_2-12y_2y_0\phi_2+\frac{4}{3}y_0^2\phi_2-\\
&-12y_0\phi_2^3+9y_0\phi_1^2\phi_2-81\phi_2^3\phi_1^2+54\phi_2^2\phi_1^3+\frac{1}{3}y_0^2\phi_1+\\
&+6y_0\phi_1\phi_2^2+27y_2^2\phi_1+54y_1\phi_2^2\phi_1-81y_2\phi_1^2\phi_2+54y_2\phi_1\phi_2^2-\\
&-6y_2y_0\phi_1-27y_1\phi_2^3+27\phi_2^4\phi_1\Big)\,. \qquad (A.1)
\end{aligned}
$$

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For ϕ_2 one has

$$0 = 3g_2(2\phi_2 - \phi_1) + 2g_6\bigg(-\frac{8}{3}y_0^2\phi_2 + 27y_1^2\phi_2 - 27y_2\phi_1 y_1 + 9y_0\phi_1 y_1 - 12y_2 y_0\phi_1 + \\ + \frac{4}{3}y_0^2\phi_1 - 36y_0\phi_1\phi_2^2 + 3y_0\phi_1^3 - 81\phi_2^2\phi_1^3 + 27\phi_2\phi_1^4 + \\ + 6y_0\phi_1^2\phi_2 + 24y_0\phi_2^3 + 54y_1\phi_2\phi_1^2 - 27y_2\phi_1^3 + 54y_2\phi_1^2\phi_2 + \\ + 24y_2 y_0\phi_2 - 81y_1\phi_2^2\phi_1 - 12y_1 y_0\phi_2 + 54\phi_2^3\phi_1^2\bigg). \tag{A.2}$$

For y_0 one finds

$$\frac{L}{y_0} = g_2 + g_6\bigg(-\frac{16}{3}y_0\phi_2^2 + \frac{4}{9}y_0^2 + \frac{8}{3}y_1 y_0 + 18y_1\phi_1\phi_2 - 24\phi_1\phi_2 y_2 - 12y_1 y_2 + \\ + \frac{16}{3}\phi_1 y_0\phi_2 - 24\phi_2^3\phi_1 + 6\phi_1^3\phi_2 + \frac{2}{3}y_0\phi_1^2 + 6\phi_2^2\phi_1^2 + 12\phi_2^4 + 12y_2^2 - \\ - 6\phi_1^2 y_2 + 24\phi_2^2 y_2 - 12y_1\phi_2^2 - \frac{16}{3}y_2 y_0\bigg). \tag{A.3}$$

For y_1 one gets

$$\frac{2L}{y_1} = 3g_2 + g_6\bigg(54y_1\phi_2^2 + \frac{4}{3}y_0^2 - 54\phi_1\phi_2 y_2 + 18\phi_1 y_0\phi_2 - 12y_2 y_0 + 54\phi_2^2\phi_1^2 - \\ - 54\phi_2^3\phi_1 - 12y_0\phi_2^2\bigg). \tag{A.4}$$

For y_2 one gets

$$\frac{L}{y_2} = 3g_2 + g_6\bigg(-54y_1\phi_1\phi_2 - 24\phi_1 y_0\phi_2 - 12y_1 y_0 + 54\phi_1^2 y_2 + 24y_2 y_0 - 54\phi_1^3\phi_2 + \\ + 54\phi_2^2\phi_1^2 - 6y_0\phi_1^2 + 24y_0\phi_2^2 - \frac{8}{3}y_0^2\bigg), \tag{A.5}$$

and finally for L we get $y_0 y_1^2 y_2 = \Lambda^8/36$.

References

[1] K.A. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality*, *Nucl. Phys.* **45BC** *(Proc. Suppl.)* (1996) 1 [hep-th/9509066].

[2] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[3] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[4] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[5] F. Cachazo and C. Vafa, *$N = 1$ and $N = 2$ geometry from fluxes*, hep-th/0206017.

JHEP09(2003)066

[6] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[7] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[8] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, *Perturbative computation of glueball superpotentials*, hep-th/0211017.

[9] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [hep-th/0211170].

[10] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, hep-th/9607163.

[11] S. Katz and C. Vafa, *Geometric engineering of $N = 1$ quantum field theories*, *Nucl. Phys.* **B 497** (1997) 196 [hep-th/9611090].

[12] C. Gomez and R. Hernandez, *M and F-theory instantons, $N = 1$ supersymmetry and fractional topological charge*, *Int. J. Mod. Phys.* **A 12** (1997) 5141 [hep-th/9701150].

[13] K.-M. Lee and P. Yi, *Monopoles and instantons on partially compactified D-branes*, *Phys. Rev.* **D 56** (1997) 3711 [hep-th/9702107].

[14] O. Aharony, A. Hanany, K.A. Intriligator, N. Seiberg and M.J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B 499** (1997) 67 [hep-th/9703110].

[15] R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, *Nonperturbative superpotentials and compactification to three dimensions*, hep-th/0304061.

[16] N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **07** (1999) 021 [hep-th/9906011];
N. Dorey, T.J. Hollowood and S. Prem Kumar, *An exact elliptic superpotential for $N = 1^*$ deformations of finite $N = 2$ gauge theories*, *Nucl. Phys.* **B 624** (2002) 95 [hep-th/0108221].

[17] R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, *Factorization of Seiberg-Witten curves and compactification to three dimensions*, hep-th/0305189.

[18] T.J. Hollowood, *Critical points of glueball superpotentials and equilibria of integrable systems*, hep-th/0305023.

[19] M. Alishahiha and A.E. Mosaffa, *On effective superpotentials and compactification to three dimensions*, *J. High Energy Phys.* **05** (2003) 064 [hep-th/0304247].

[20] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Integrability and Seiberg-Witten exact solution*, *Phys. Lett.* **B 355** (1995) 466 [hep-th/9505035].

[21] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, *Nucl. Phys.* **B 459** (1996) 97 [hep-th/9509161].

[22] T. Nakatsu and K. Takasaki, *Whitham-Toda hierarchy and $N = 2$ supersymmetric Yang-Mills theory*, *Mod. Phys. Lett.* **A 11** (1996) 157 [hep-th/9509162].

[23] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[24] T. Eguchi and S.-K. Yang, *Prepotentials of $N = 2$ supersymmetric gauge theories and soliton equations*, *Mod. Phys. Lett.* **A 11** (1996) 131 [hep-th/9510183].

JHEP09(2003)066

[25] H. Itoyama and A. Morozov, *Integrability and Seiberg-Witten theory: curves and periods*, *Nucl. Phys.* **B 477** (1996) 855 [hep-th/9511126].

[26] E. D'Hoker and D.H. Phong, *Lectures on supersymmetric Yang-Mills theory and integrable systems*, hep-th/9912271.

[27] A. Mironov, *Seiberg-Witten theories, integrable models and perturbative prepotentials*, hep-th/0010078.

[28] H. Itoyama and A. Morozov, *Calculating gluino condensate prepotential*, *Prog. Theor. Phys.* **109** (2003) 433 [hep-th/0212032]; *Gluino-condensate (CIV-DV) prepotential from its Whitham-time derivatives*, hep-th/0301136.

[29] M. Aganagic, K. Intriligator, C. Vafa and N.P. Warner, *The glueball superpotential*, hep-th/0304271.

[30] K. Landsteiner, J.M. Pierre and S.B. Giddings, *On the moduli space of $N = 2$ supersymmetric G_2 gauge theory*, *Phys. Rev.* **D 55** (1997) 2367 [hep-th/9609059].

[31] I. Pesando, *Exact results for the supersymmetric G_2 gauge theories*, *Mod. Phys. Lett.* **A 10** (1995) 1871 [hep-th/9506139];
S.B. Giddings and J.M. Pierre, *Some exact results in supersymmetric theories based on exceptional groups*, *Phys. Rev.* **D 52** (1995) 6065 [hep-th/9506196];
P. Pouliot, *Chiral duals of nonchiral SUSY gauge theories*, *Phys. Lett.* **B 359** (1995) 108 [hep-th/9507018].

[32] A. Brandhuber, H. Ita, H. Nieder, Y. Oz and C. Romelsberger, *Chiral rings, superpotentials and the vacuum structure of $N = 1$ supersymmetric gauge theories*, hep-th/0303001.

[33] N.Ja. Vilenkin and A.U. Klimyk, *Representation of Lie groups and special functions*, vol. 3, Kluwer 1992.

[34] P.C. Argyres and M.R. Douglas, *New phenomena in* SU(3) *supersymmetric gauge theory*, *Nucl. Phys.* **B 448** (1995) 93 [hep-th/9505062].

[35] P.C. Argyres, M. Ronen Plesser, N. Seiberg and E. Witten, *New $N = 2$ superconformal field theories in four dimensions*, *Nucl. Phys.* **B 461** (1996) 71 [hep-th/9511154];
T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $N = 2$ superconformal field theories in* 4 *dimensions*, *Nucl. Phys.* **B 471** (1996) 430 [hep-th/9603002].

[36] P. van Moerbeke and D. Mumford, *The spectrum of difference operators and algebraic curves*, *Acta Math.* **143** 1-2 (1979) 93.

[37] U.H. Danielsson and B. Sundborg, *Exceptional equivalences in $N = 2$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 370** (1996) 83 [hep-th/9511180];
M. Alishahiha, F. Ardalan and F. Mansouri, *The moduli space of the supersymmetric g_2 Yang-Mills theory*, *Phys. Lett.* **B 381** (1996) 446 [hep-th/9512005];
M.R. Abolhasani, M. Alishahiha and A.M. Ghezelbash, *The moduli space and monodromies of the $N = 2$ supersymmetric Yang-Mills theory with any Lie gauge groups*, *Nucl. Phys.* **B 480** (1996) 279 [hep-th/9606043].

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